MATH 108 Fall 2019 - **Problem Set 10**

due December 6

1. (a) Given that $G = \{e, u, v, w\}$ is a group of order 4 with identity e, $u^2 = v$ and $v^2 = e$, construct the operation table for G.

   (b) Given that $H = \{a, b, c, d\}$ is a group of order 4 with identity a and $b^2 = c^2 = d^2 = a$, construct the operation table for H.

2. Find all subgroups of the symmetric group on three elements, $\mathfrak{S}_3$.

3. The *dihedreal group of the square*, D_4, is the group of the symmetries of a square. Let $e \in D_4$ be the identity element. Let $r \in D_4$ denote a $90°$ counter-clockwise rotation of the square. Let $s \in D_4$ denote a reflection of the square across a vertical line through the center. List the eight elements of D_4 in terms of r and s and find the order of each element. (You can physically model D_4 by rotating and flipping a square of paper.)

4. Let G be a group (represented multiplicatively) and let $f : G \to G$ be the function defined by $f(x) = x^{-1}$. Prove that f is a group homomorphism if and only if G is abelian.

5. For each pair of groups, demonstrate an isomorphism between them or prove that they are not isomorphic.

   (a) $(\mathbb{Z}/4\mathbb{Z}, +)$ and $(\{1, -1, i, -i\}, \cdot)$.

   (b) $\mathfrak{S}_3$ and $(\mathbb{Z}/6\mathbb{Z}, +)$.

   (c) G and H defined in Problem 1.

   (d) $(\mathbb{Z}/5\mathbb{Z} \setminus \{\overline{0}\}, \cdot)$ and $(\mathbb{Z}/4\mathbb{Z}, +)$.

6. Let G and H be groups with e the identity element of H. For group homomorphism $f : G \to H$, the *kernel* of f, denoted $\ker(f)$, is defined as

   $$\ker(f) = \{g \in G \mid f(g) = e\}.$$

   Prove that $\ker(f)$ is a subgroup of G.

7. Let G be a finite group (represented multiplicatively) and H a subgroup of G. Define a relation $\sim$ on G by $a \sim b$ if and only if $ab^{-1} \in H$.

   (a) Prove that $\sim$ is an equivalence relation.

   (b) Prove that every equivalence class of $\sim$ has cardinality $|H|$.

   (c) Prove that $|H|$ divides $|G|$.