# MATH 108 Fall 2019 - **Problem Set 8 solutions**

due November 22

1. Let $X, Y, Z, W$ be sets with $|X| = |Z|$ and $|Y| = |W|$.

   (a) Cardinal addition is defined by $|X| + |Y| = |X \cup Y|$ where $X$ and $Y$ are disjoint. Prove that cardinal addition is well-defined, meaning that

   $$|X| + |Y| = |Z| + |W|$$

   where $X$ and $Y$ are disjoint and $Z$ and $W$ are disjoint.

   Since $|X| = |Z|$ and $|Y| = |W|$, there are bijections $f : X \to Z$ and $g : Y \to W$. We define $h : X \cup Y \to Z \cup W$ by letting $h(x) = f(x)$ if $x \in X$ and $h(y) = g(y)$ if $y \in Y$. For each $z \in Z$, then there is a unique $x \in X$ such that $h(x) = z$ and there is no $y \in Y$ with $h(y) = z$ since $h(y) \in W$ for all $y \in Y$ and $Z$ and $W$ are disjoint. Similarly if $w \in W$ then there is a unique $y \in Y$ such that $h(y) = w$ and no $x \in X$. This proves that $h$ is a bijection. So

   $$|X \cup Y| = |Z \cup W|.$$

   (b) Cardinal multiplication is defined by $|X| \cdot |Y| = |X \times Y|$. Prove that cardinal multiplication is well-defined, meaning that

   $$|X| \cdot |Y| = |Z| \cdot |W|.$$

   With $f, g$ as in part (a), define $h : X \times Y \to Z \times W$ by $h(x, y) = (f(x), g(y))$. For each $(z, w) \in Z \times W$, there is a unique $x \in X$ such that $f(x) = z$ and a unique $y \in Y$ such that $f(y) = w$, so $(x, y)$ is the unique pair with $h(x, y) = (z, w)$. This proves that $h$ is a bijection. So

   $$|X \times Y| = |Z \times W|.$$

   (c) Cardinal exponentiation is defined by $2^{|X|} = |\mathcal{P}(X)|$. Prove that cardinal exponentiation is well-defined, meaning that

   $$2^{|X|} = 2^{|Z|}.$$

   With $f$ as in part (a), define $h : \mathcal{P}(X) \to \mathcal{P}(Z)$ by $h(A) = \{f(x) \mid x \in A\}$. To prove that $h$ is injective, suppose that $h(A) = h(B)$ for some sets $A, B \subseteq X$. For $x \in A$, $f(x) \in h(A) = h(B)$ so there is $y \in B$ such that $f(y) = f(x)$. Since $f$ is injective, $x = y$, so $x \in B$. This proves that $A \subseteq B$. By the same argument, $B \subseteq A$ so $A = B$. To prove $h$ is surjective, for $C \subseteq Z$, let $A = \{x \in X \mid f(x) \in C\}$. For each $z \in C$, there is $x \in A$ such that $f(x) = z$ since $f$ is surjective. Therefore $h(A) = C$, so $h$ is surjective. So

   $$|\mathcal{P}(X)| = |\mathcal{P}(Z)|.$$

2. Let $n$ be a positive integer. Prove that the set of positive integer divisors of $n$ is finite.

   Let $A$ be the set of divisors of $n$. If $a \in A$ then $a \leq n$. Therefore $A \subseteq \{1, 2, \ldots, n\}$. The set $\{1, 2, \ldots, n\}$ has cardinality $n$ and any subset of a finite set is finite, so $A$ is finite.

3. (a) Prove that $|\{x \in \mathbb{R} \mid -1 < x < 1\}| = |\mathbb{R}|$.

   In Problem Set 7 you were asked to find a bijection between $\{x \in \mathbb{R} \mid -1 < x < 1\}$ and $\mathbb{R}$, for example $f(x) = \frac{1}{x+1} + \frac{1}{x-1}$. Therefore these sets have the same cardinality.

   (b) Prove that $|\{x \in \mathbb{R} \mid -1 \leq x \leq 1\}| = |\mathbb{R}|$.

   We have the following set containments

   $$\{x \in \mathbb{R} \mid -1 < x < 1\} \subseteq \{x \in \mathbb{R} \mid -1 \leq x \leq 1\} \subseteq \mathbb{R}.$$

   Therefore we get the relations

   $$|\{x \in \mathbb{R} \mid -1 < x < 1\}| \leq |\{x \in \mathbb{R} \mid -1 \leq x \leq 1\}| \leq |\mathbb{R}|.$$

   Since the left and right most cardinalities are equal to $|\mathbb{R}|$ by part (a), the middle cardinality is also equal to $|\mathbb{R}|$ by the Cantor-Schröder-Bernstein Theorem.

4. Prove each of the following sets is countable.

   (a) The set of prime numbers.

   The set of prime numbers is a subset of $\mathbb{Z}$, which is countable. A subset of a countable set is countable.

   (b) $\mathbb{Z} \times \mathbb{Z}$.

   $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{Z}| \cdot |\mathbb{Z}| = \aleph_0 \cdot \aleph_0 = \aleph_0$.

   (c) The set of all finite-length binary strings, $\bigcup_{n=0}^{\infty} \{0, 1\}^n$. (This is the set of all possible computer files.)

   For each $n \in \mathbb{Z}_{\geq 0}$, the set $\{0, 1\}^n$ of binary strings of length $n$ is finite (with size $2^n$). Therefore we can enumerate all finite-length binary strings by first listing all strings of length 0, then all strings of length 1, then all strings of length 2, etc:

   $$(), 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 0000, \ldots$$

   This enumeration gives a bijection from $\mathbb{Z}_{>0}$ to $\bigcup_{n=0}^{\infty} \{0, 1\}^n$, so it is countable.

   Alternate proof: Each set $\{0, 1\}^n$ is finite, with cardinality $2^n < \aleph_0$, so there is an injective map $h_n : \{0, 1\}^n \to \mathbb{Z}_{\geq 0}$. A concrete way to define $h_n$ is to send each binary string to the number it represents (e.g. $h_4(0101) = 5$). Then define function $h : \bigcup_{n=0}^{\infty} \{0, 1\}^n \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ by $h(a) = (n, h_n(a))$ for $a \in \{0, 1\}^n$. It's clear from construction that $h$ is injective so

   $$\left| \bigcup_{n=0}^{\infty} \{0, 1\}^n \right| \leq |\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}| = \aleph_0.$$

5. Prove that the set of irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$, is uncountable.

   We proceed by contradiction. Suppose $\mathbb{R} \setminus \mathbb{Q}$ is countable. We know that $\mathbb{Q}$ is countable and that the union of two countable sets is countable. Therefore

   $$\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q}$$

   is also countable. But this contradicts the fact that $\mathbb{R}$ is uncountable. Therefore $\mathbb{R} \setminus \mathbb{Q}$ is uncountable.

6. Use Cantor's diagonalization argument to prove that the set of all functions from $\mathbb{Z}_{>0}$ to $\mathbb{Z}_{>0}$ is uncountable.

   We proceed by contradiction, so assume that the set $A$ of functions $\mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is countable. Then there is a surjective function $f : \mathbb{Z}_{>0} \to A$. For each $n \in \mathbb{Z}_{>0}$, we get a function $f(n) : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ in $A$. We will construct a new function $g \in A$ that is not among those in the image of $f$. For each $n \in \mathbb{Z}_{>0}$, choose $g(n)$ so that $g(n) \neq (f(n))(n)$. Then $g \neq f(n)$ for all $n \in \mathbb{Z}_{>0}$. But this contradicts the fact that $f$ is surjective. Therefore no such surjective function $f$ can exist, so $A$ is uncountable.

7. Let $X$ be an infinte set.

   (a) Prove that $|X| \geq \aleph_0$.

   $\mathcal{P}(X) \setminus \{\emptyset\}$ is a collection of nonempty sets, so by the Axiom of Choice there is a choice function $c : \mathcal{P}(X) \setminus \{\emptyset\} \to X$ with $c(B) \in B$. Define function $f : \mathbb{Z}_{>0} \to X$ as follows. For each $n \in \mathbb{Z}_{>0}$, let $f(n) = c(X \setminus \{f(1), \ldots, f(n-1)\})$. Each set $X \setminus \{f(1), \ldots, f(n-1)\}$ is non-empty since $X$ is infinite. The resulting function $f$ is injective since each value is chosen to be distinct from the previous ones. Therefore $|\mathbb{Z}_{>0}| \leq |X|$.

   (b) Prove that $|X| + 1 = |X|$.

   [Hint: First prove it for the case that $X$ is countably infinite. Then for the general case, part (a) implies that $X$ has a countably infinite subset $Y$. Use the fact that $|Y| + 1 = |Y|$.]

   First consider the case that $X = \mathbb{Z}_{>0}$. The set $\mathbb{Z}_{\geq 0}$ is equal to the disjoint union $\mathbb{Z}_{>0} \cup \{0\}$ so $|\mathbb{Z}_{\geq 0}| = |X| + 1$. The bijection $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{>0}$ defined by $g(x) = x + 1$ demonstrates that $|\mathbb{Z}_{\geq 0}| = |\mathbb{Z}_{>0}|$ so $|X| + 1 = |X|$. We also proved this in class (Hilbert's Grand Hotel).

   Now consider the general case of infinite set $X$. By part (a), there is an injective function $f : \mathbb{Z}_{>0} \to X$. Let $Y \subseteq X$ be the image of $f$. Since $|Y| = |\mathbb{Z}_{>0}|$, we have $|Y| + 1 = |Y|$. The sets $X \setminus Y$ and $Y$ are disjoint and $X = (X \setminus Y) \cup Y$ so

   $$|X| + 1 = |X \setminus Y| + |Y| + 1 = |X \setminus Y| + |Y| = |X|.$$