MATH 108 Fall 2019 - **Problem Set 9 solutions**

due December 2

1. Prove that $\mathfrak{c} + \mathfrak{c} = \mathfrak{c}$ (where $\mathfrak{c} = |\mathbb{R}|$, the cardinality of the continuum).

   We know that $|\mathbb{R}| = |(0,1)| = \mathfrak{c}$. In addition, $|(1,2)| = \mathfrak{c}$ since there is bijection $f : (0,1) \to (1,2)$ defined by $f(x) = x + 1$. Since intervals $(0,1)$ and $(1,2)$ are disjoint,

   $$\mathfrak{c} + \mathfrak{c} = |(0,1) \cup (1,2)|.$$

   By the containment relations

   $$(0,1) \subseteq (0,1) \cup (1,2) \subseteq \mathbb{R},$$

   $$\mathfrak{c} = |(0,1)| \leq |(0,1) \cup (1,2)| \leq |\mathbb{R}| = \mathfrak{c}.$$

   Therefore $\mathfrak{c} + \mathfrak{c} = \mathfrak{c}$.

2. Order the following cardinalities: $|(0,1)|$, $|[0,1]|$, $|\{0,1\}|$, $|\{0\}|$, $|\mathcal{P}(\mathbb{R})|$, $|\mathbb{Q}|$, $|\emptyset|$, $|\mathbb{R}^2|$, $|\mathcal{P}(\mathcal{P}(\mathbb{R}))|$, $|\mathbb{R}|$, $|\mathcal{P}(\mathbb{Q})|$.

   $|\emptyset| < |\{0\}| < |\{0,1\}| < |\mathbb{Q}| < |\mathbb{R}| = |(0,1)| = |[0,1]| = |\mathcal{P}(\mathbb{Q})| = |\mathbb{R}^2| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))|$.

3. Determine whether each algebraic structure is a group. If no, which properties does it fail? If yes, is it abelian? Find an identity element if one exists.

   (a) $(\mathbb{Z}_{>0}, +)$.
   Not a group. $\mathbb{Z}_{>0}$ has no additive identity, and elements have no additive inverses. There are no identity elements.

   (b) $(\mathbb{Q}, \cdot)$.
   Not a group. 0 has no multiplicative inverse. The identity element is 1.

   (c) $(\mathbb{Z}/4\mathbb{Z}, +)$.
   Group. It is abelian because $\overline{x} + \overline{y} = \overline{x+y} = \overline{y} + \overline{x}$. The identity element is $\overline{0}$.

   (d) $(\mathbb{Z}/4\mathbb{Z} \setminus \{\overline{0}\}, \cdot)$.
   Not a group. Not even an algebraic structure, because $\overline{2} \cdot \overline{2} = \overline{0}$ is not in the set $\mathbb{Z}/4\mathbb{Z} \setminus \{\overline{0}\}$. Therefore the operation $\cdot$ is not well-defined.

   (e) (The set of functions $\{1,2,3\} \to \{1,2,3\}$, composition).
   Not a group. If function f is not injective, then it has no inverse. The identity element is the identity function $I_{\{1,2,3\}}$.

   (f) (The set of bijective functions $\{1,2,3\} \to \{1,2,3\}$, composition).
   Group. It is not abelian. For example let f be the function that switches 1 and 2, and g be the function that switches 2 and 3. Then $g \circ f(1) = 3$ but $f \circ g(1) = 2$. The identity element is the identity function $I_{\{1,2,3\}}$.

(g) (The set of 2×2 real matrices with determininant 1, matrix multiplication).

Group. It is not abelian. For example let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

The identity element is the identity matrix,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

4. Write the Cayley table for the following finite algebraic structures.

(a) $(\mathbb{Z}/4\mathbb{Z}, +)$.

$+$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{0}$
$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{0}$	$\overline{1}$
$\overline{3}$	$\overline{3}$	$\overline{0}$	$\overline{1}$	$\overline{2}$

(b) $(\mathbb{Z}/4\mathbb{Z}, \cdot)$.

$\cdot$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{0}$	$\overline{2}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

(c) (The set of bijective functions $\{1, 2, 3\} \to \{1, 2, 3\}$, composition).

Representing each bijection $f : \{1, 2, 3\} \to \{1, 2, 3\}$ by its sequence of values, $(f(1), f(2), f(3))$,

$\circ$	$(1,2,3)$	$(1,3,2)$	$(2,1,3)$	$(2,3,1)$	$(3,1,2)$	$(3,2,1)$
$(1,2,3)$	$(1,2,3)$	$(1,3,2)$	$(2,1,3)$	$(2,3,1)$	$(3,1,2)$	$(3,2,1)$
$(1,3,2)$	$(1,3,2)$	$(1,2,3)$	$(3,1,2)$	$(3,2,1)$	$(2,1,3)$	$(2,3,1)$
$(2,1,3)$	$(2,1,3)$	$(2,3,1)$	$(1,2,3)$	$(1,3,2)$	$(3,2,1)$	$(3,1,2)$
$(2,3,1)$	$(2,3,1)$	$(2,1,3)$	$(3,2,1)$	$(3,1,2)$	$(1,2,3)$	$(1,3,2)$
$(3,1,2)$	$(3,1,2)$	$(3,2,1)$	$(1,3,2)$	$(1,2,3)$	$(2,3,1)$	$(2,1,3)$
$(3,2,1)$	$(3,2,1)$	$(3,1,2)$	$(2,3,1)$	$(2,1,3)$	$(1,3,2)$	$(1,2,3)$