MATH 108 Winter 2019 - **Problem Set 3**

due February 1

1. Using induction, prove that for all positive integers n,
   (a) $n^3 - n$ is divisible by 3.
   (b) $8^n - 1$ is divisible by 7.
   (c) $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.
   (d) $n! = 1 + \sum_{k=1}^{n-1} k \cdot k!$.

2. In American football, a team can score seven points for each touchdown, and three points for each field goal (ignore safeties, two-point conversions, etc). Prove that every integer score larger than 11 is possible.

3. Let P be the set of prime numbers. Prove that

$$\bigcup_{p \in P} p\mathbb{Z} = \mathbb{Z} \setminus \{-1, 1\}.$$

4. Using induction, prove that if A is a finite set with $|A| = n$ then $|\mathcal{P}(A)| = 2^n$.

5. Use the Well-Ordering Principle of the natural numbers to prove that every positive rational number x can be expressed as a fraction $x = a/b$ where a and b are postive integers with no common factor.

6. The Fibonacci sequence is an infinite sequence of integers $(f_0, f_1, f_2, f_3, \ldots)$ defined as follows. The first two numbers are $f_0 = 0$ and $f_1 = 1$. For all $n \geq 2$, define f_n to be the sum of the previous two numbers,

$$f_n = f_{n-1} + f_{n-2}.$$

   Use induction to prove that for all nonnegative integers n,

$$f_n = \frac{\varphi^n - \psi^n}{\varphi - \psi},$$

   where $\varphi = (1 + \sqrt{5})/2$ and $\psi = (1 - \sqrt{5})/2$.

7. For each positive integer n, let T_n count the number of ways to tile a $2 \times n$ rectangle using 2×1 tiles. Prove for each positive integer n that $T_n = f_{n+1}$, the $(n+1)$th Fibonacci number.