MATH 108 Winter 2019 - **Problem Set 8**

due March 15

1. (a) Given that $G = \{e, u, v, w\}$ is a group of order 4 with identity e, $u^2 = v$ and $v^2 = e$, construct the operation table for G.
   (b) Given that $H = \{a, b, c, d\}$ is a group of order 4 with identity a and $b^2 = c^2 = d^2 = a$, construct the operation table for H.

2. Find all subgroups of the symmetric group on three elements, $\mathfrak{S}_3$.

3. Let G be the symmetry group of a square. Let $e \in G$ be the identity element. Let $r \in G$ denote a $90°$ counter-clockwise rotation of the square. Let $s \in G$ denote a reflection of the square across a vertical line through the center. List the eight elements of G in terms of r and s and find the order of each element. (You can physically model G by rotating and flipping a square of paper.)

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$.
   (a) Is $f : (\mathbb{R}, +) \to (\mathbb{R}, +)$ a group homomorphism? Justify your answer.
   (b) Is $f : (\mathbb{R} \setminus \{0\}, \cdot) \to (\mathbb{R} \setminus \{0\}, \cdot)$ a group homomorphism? Justify your answer.

5. Let G be a group (represented multiplicatively) and H a subgroup of G. Define a relation $\sim$ on G by $a \sim b$ if and only if $ab^{-1} \in H$.
   (a) Prove that $\sim$ is an equivalence relation.
   (b) Suppose that G is finite. Prove that every equivalence class of $\sim$ has size $|H|$. Conclude that $|H|$ divides $|G|$.

6. For each pair of groups, demonstrate an isomorphism between them or prove that they are not isomorphic.
   (a) $(\mathbb{Z}/4\mathbb{Z}, +)$ and $(\{1, -1, i, -i\}, \cdot)$.
   (b) $\mathfrak{S}_3$ and $(\mathbb{Z}/6\mathbb{Z}, +)$.
   (c) G and H defined in Problem 1.
   (d) $(\mathbb{Z}/7\mathbb{Z} \setminus \{\overline{0}\}, \cdot)$ and $(\mathbb{Z}/6\mathbb{Z}, +)$.

7. Let G and H be groups with e the identity element of H. For group homomorphism $f : G \to H$, the *kernel* of f, denoted $\ker(f)$, is defined as

$$\ker(f) = \{g \in G \mid f(g) = e\}.$$

   Prove that $\ker(f)$ is a subgroup of G.